Chaotic Dynamical Systems

Michael Grantham
Mentor: James O’Quinn

Texas A&M University

23 November 2020
Talk Outline

- Dynamical Systems Background
- Hyperbolicity
- The Schwarzian Derivative
- Chaotic Systems
- Bifurcation
Motivation

- Imagine you want to model the population of some species.
- Have some function $f(x)$ where $x$ is the current population.
- Then for time $n$, $f^n(x) = f \circ f \circ \cdots \circ f(x)$.
- This is a Dynamical System.

**Definition**

A **Dynamical System**, $(X, T)$ is comprised of a space $X$ and some function/map $T$ on $X$.

**Logistic Map**

The **Logistic Map**, $F_\mu = \mu x(1 - x)$ is dynamical system on $\mathbb{R}$. One of its applications is population modeling.
Dynamical Systems

**Definition**

The **orbit** of a point is the path it takes through its iterations, \( \{ f^n(x) \mid n \in 0, 1, 2, \ldots \} \)

**Definition**

A point is a **fixed point** if for \( f \), \( f(x) = x \). If for some \( n \), \( f^n(x) = x \) then \( x \) is a **periodic point** of period \( n \).
Definition

Let $p$ be a periodic point of period $n$. A point $x$ is **forward asymptotic** to $p$ if \( \lim_{i \to \infty} f^i(x) = p \).

Definition

The **stable set** $W_s(p)$, consists of all points which are forward asymptotic to $p$.

Example

For $F_2$, all points on the unit interval are forward asymptotic to $1/2$, Therefore the stable set is: $W_s(1/2) = (0, 1)$
Hyperbolicity

Definition
Let $p$ be a periodic point of period $n$. The point $p$ is **hyperbolic** if $|(f^n)'(p)| \neq 1$.

Definition
Let $p$ be a hyperbolic point of period $n$. If $|(f^n)'(p)| < 1$ then $p$ is an **attractor** (attracting fixed point, or a sink).

Definition
Let $p$ be a fixed point. If $|f'(p)| > 1$ then $p$ is a **repeller** (repelling fixed point, or source).
Example: Take $F_{\mu}$, $\mu = 2$ as before. $x = 1/2$ is a hyperbolic attractor.
Definition

The **Schwarzian Derivative** of a function $f$ is

$$Sf(x) = \frac{f''''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f'''(x)}{f'(x)} \right)^2$$

Theorem

If $Sf < 0$ and $f$ has $n$ critical points. Then $f$ has at most $n + 2$ attracting periodic orbits.
Schwarzian Derivative

Intuition

- A bounded stable set must have a critical point.
- There are at most 2 unbounded stable sets.
- Therefore the upper bound on number of stable sets, and therefore attracting points, is $n + 2$. 

\[ (\leftarrow \right) \]
- **Topological transitivity**
  - For some $f : J \to J$ if any pair of open sets $U, V \subset J$, $\exists k > 0$ s.t. $f^k(U) \cap V \neq \emptyset$.
  - This is equivalent to having a dense orbit.

- **Sensitivity to initial conditions**
  - $\exists \delta > 0$ s.t. $\forall x \in X$ and every open set $U$ which contains $x$ $\exists y \in U$ s.t. $|f^n(x) - f^n(y)| > \delta$.
  - Butterfly Effect

- **Dense periodic points**
Take $F_\mu$ with $\mu = 2, 4$ on the unit interval.

- $F_2$ is not chaotic.
  - Recall that $W_s(1/2)$ is the unit interval. Therefore sensitivity to initial conditions isn’t satisfied.
  - Note: topological transitivity is not satisfied and this does not have dense periodic points either.
  - This is true for all $\mu < 4$

- $F_4$ is chaotic.
  - Observe visually that the shown orbit is dense.
  - Observe that for any two points their orbits get farther apart.
  - Periodic points are dense.