Chaotic Dynamical Systems

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Talk Outline

- Dynamical Systems Background
- Hyperbolicity
- The Schwarzian Derivative
- Chaotic Systems
- Bifurcation

Motivation

- Imagine you want to model the population of some species.
- Have some function $f(x)$ where x is the current population.
- Then for time n, $f^n(x) = f \circ f \circ \cdots \circ f(x)$.
- This is a **Dynamical System**.

Definition

A **Dynamical System**, (X, T) is comprised of a space X and some function/map T on X.

Logistic Map

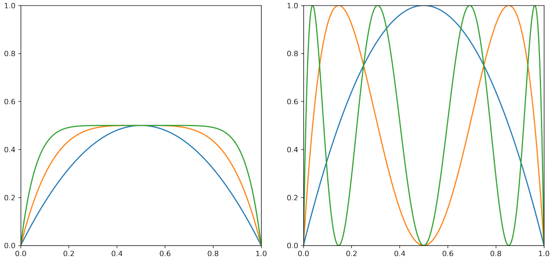
The **Logistic Map**, $F_\mu = \mu x(1 - x)$ is dynamical system on $\mathbb{R}$. One of its applications is population modeling.

Definition

The **orbit** of a point is the path it takes through it's iterations, $\{f^n(x) \mid n \in 0, 1, 2, \dots\}$

Definition

A point is a **fixed point** if for f, $f(x) = x$. If for some n, $f^n(x) = x$ then x is a **periodic point** of period n



Dynamical Systems

Definition

Let p be a periodic point of period n. A point x is **forward asymptotic** to p if $\lim_{i\to\infty} f^{in}(x) = p$.

Definition

The **stable set**, $W_s(p)$, consists of all points which are forward asymptotic to p.

Example

For F_2, all points on the unit interval are forward asymptotic to $1/2$,
Therefore the stable set is: $W_s(1/2) \ = \ (0, 1)$

Hyperbolicity

Definition

Let p be a perioidc point of period n. The point p is **hyperbolic** if $|(f^n)'(p)| \neq 1$.

Definition

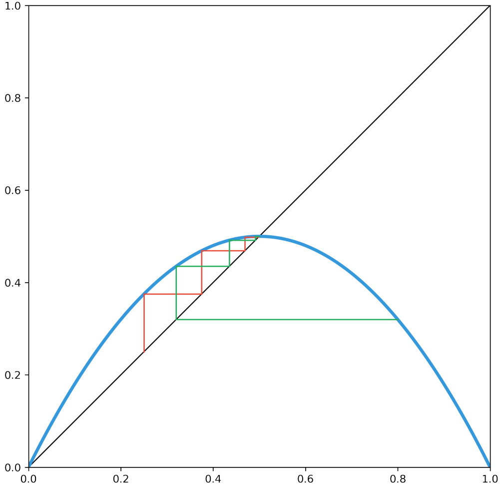
Let p be a hyperbolic point of period n. If $|(f^n)'(p)| < 1$ then p is an **attractor** (attracting fixed point, or a sink).

Definition

Let p be a fixed point. If $|f'(p)| > 1$ then p is a **repellor** (repelling fixed point, or source).

Hyperbolicity

Example: Take F_μ, $\mu = 2$ as before. $x = 1/2$ is a hyperbolic attractor.



Schwarzian Derivative

Definition

The **Schwarzian Derivative** of a function f is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Theorem

If $Sf < 0$ and f has n critical points. Then f has at most $n+2$ attracting periodic orbits.

Schwarzian Derivative

Intuition

- A bounded stable set must have a critical point.
- There are at most 2 unbounded stable sets.
- Therefore the upper bound on number of stable sets, and therefore attracting points, is $n + 2$.

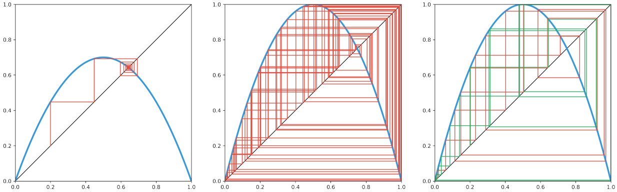
Chaos

- Topological transitivity
  - For some $f : J \to J$ if any pair of open sets $U, V \subset J$, $\exists k > 0$ s.t. $f^k(U) \cap V \neq \emptyset$.
  - This is equivalent to having a dense orbit.
- Sensitivity to initial conditions
  - $\exists \delta > 0$ s.t. $\forall x \in X$ and every open set U which contains x $\exists y \in U$ s.t. $|f^n(x) - f^n(y)| > \delta$.
  - Butterfly Effect
- Dense periodic points

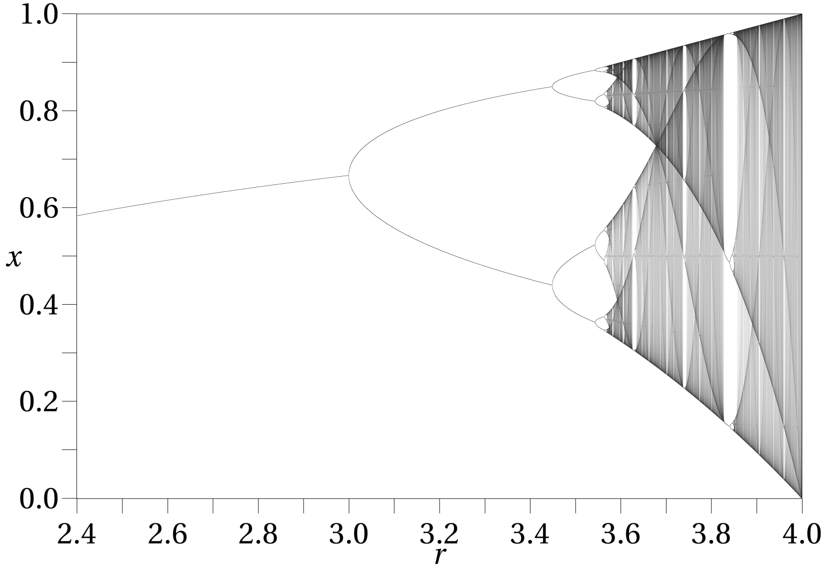
Chaos

Take F_μ with $\mu = 2, 4$ on the unit interval.

- F_2 is not chaotic.
  - Recall that $W_s(1/2)$ is the unit interval. Therefore sensitivity to initial conditions isn't satisfied.
  - Note: topological transitivity is not satisfied and this does not have dense periodic points either.
  - This is true for all $\mu < 4$
- F_4 is chaotic.
  - Observe visually that the shown orbit is dense.
  - Observe that for any two points their orbits get farther apart.
  - Periodic points are dense.



Bifurcation



Bifurcation Diagram of the Logistic Map by Jordan Pierce, CC0, via Wikimedia Commons

References

Devaney, Robert L. *An Introduction to Chaotic Dynamical Systems*. 2nd Ed. (1989).